

ERRATA FOR : TOPOLOGICAL COMPLEXITY IS A FIBREWISE L-S CATEGORY

NORIO IWASE[†] AND MICHIMIRO SAKAI

ABSTRACT. There is a problem with the proof of Theorem 1.13 of [2] which states that for a fibrewise well-pointed space X over B , we have $\text{cat}_B^B(X) = \text{cat}_B^*(X)$ and that for a locally finite simplicial complex B , we have $\mathcal{TC}(B) = \mathcal{TC}^M(B)$. While we still conjecture that Theorem 1.13 is true, this problem means that, at present, no proof is given to exist. Alternatively, we show the difference between two invariants $\text{cat}_B^*(X)$ and $\text{cat}_B^B(X)$ is at most 1 and the conjecture is true for some cases. We give further corrections mainly in the proof of Theorem 1.12.

It was pointed out to the authors by Jose Calcines that there is a problem with the proof of Theorem 1.13 of [2] which states that for a fibrewise well-pointed space X over B , we have $\text{cat}_B^B(X) = \text{cat}_B^*(X)$ and that for a locally finite simplicial complex B , we have $\mathcal{TC}(B) = \mathcal{TC}^M(B)$, where $\text{cat}_B^*(X)$ and $\mathcal{TC}^M(B)$ are new versions of a fibrewise L-S category and a topological complexity, respectively, which are introduced in [2].

While we still conjecture that Theorem 1.13 of [2] is true, this problem means that, at present, no proof is given to exist. It then results that “ $\mathcal{TC}(B)$ ” in Corollary 8.7 of [2] must be replaced with “ $\mathcal{TC}^M(B)$ ” and the resulting inequality should be presented in the following form:

$$\mathcal{Z}_\pi(B) \leq \text{wgt}_\pi(B) \leq \text{Mwgt}_B^B(d(B)) \leq \mathcal{TC}^M(B) - 1 \leq \text{catlen}_B^B(d(B)) \leq \text{Cat}_B^B(d(B)).$$

The problem in the argument occurs on page 14 where a homotopy

$$\hat{\Phi}_i : \hat{U}_i \times [0, 1] \rightarrow \hat{X}$$

is given, while the definition of $\hat{\Phi}_i$ apparently is not well-defined. Alternatively, we show here the difference between two invariants $\text{cat}_B^*(X)$ and $\text{cat}_B^B(X)$ is at most 1 and the conjecture is true for some cases.

Theorem 1. *For a fibrewise well-pointed space X over B , we have $\text{cat}_B^*(X) \leq \text{cat}_B^B(X) \leq \text{cat}_B^*(X) + 1$ which implies that, for a locally finite simplicial complex B , we have $\mathcal{TC}(B) \leq \mathcal{TC}^M(B) \leq \mathcal{TC}(B) + 1$.*

Proof: The inequality of $\mathcal{TC}(B)$ and $\mathcal{TC}^M(B)$ in Theorem 1 for a locally finite simplicial complex B is, by Theorem 1.7 in [2], a special case of the inequality of $\text{cat}_B^*(X)$ and $\text{cat}_B^B(X)$ in Theorem 1 for a fibrewise well-pointed space X . So it is

Date: February 28, 2012.

2000 Mathematics Subject Classification. Primary 55M30, Secondary 55Q25.

Key words and phrases. Topological complexity, Lusternik-Schnirelmann category.

[†] supported by the Grant-in-Aid for Scientific Research #22340014 from Japan Society for the Promotion of Science.

sufficient to show the inequality for X : because the inequality $\text{cat}_B^*(X) \leq \text{cat}_B^B(X)$ is clear by definition, all we need to show is the inequality $\text{cat}_B^B(X) \leq \text{cat}_B^*(X) + 1$. Let X be a fibrewise well-pointed space over B with a projection $p_X : X \rightarrow B$ and a section $s_X : B \rightarrow X$. Let (u, h) be a fibrewise (strong) Strøm structure (see Crabb and James [1]) on $(X, s_X(B))$, i.e., $u : X \rightarrow [0, 1]$ is a map and $h : X \times [0, 1] \rightarrow X$ is a fibrewise pointed homotopy such that $u^{-1}(0) = s_X(B)$, $h(x, 0) = x$ for any $x \in X$ and $h(x, 1) = s_X \circ p_X(x)$ for any $x \in X$ with $u(x) < 1$. Assume $\text{cat}_B^*(X) = m$ and the family $\{U_i; 0 \leq i \leq m\}$ of open sets of X satisfies $X = \bigcup_{i=0}^m U_i$ and each open set U_i is fibrewise contractible (into $s_X(B)$) by a fibrewise homotopy $H_i : U_i \times [0, 1] \rightarrow X$. Let $V_i = U'_i \cup V$ for $0 \leq i \leq m$ and $V_{m+1} = u^{-1}([0, \frac{2}{3}))$ where $U'_i = U_i \setminus u^{-1}([0, \frac{1}{2}])$ and $V = u^{-1}([0, \frac{1}{3}))$. Then the restriction $H_i|_{U'_i} : U'_i \times [0, 1] \rightarrow X$ gives a fibrewise contraction of U'_i and the restriction of the fibrewise (strong) Strøm structure $h|_V : V \times [0, 1] \rightarrow X$ gives a fibrewise pointed contraction of V . Since U'_i and V are obviously disjoint, we obtain that $V_i = U'_i \cup V \supset \Delta(B)$ is a fibrewise contractible open set by a fibrewise pointed homotopy. Similarly the restriction of the fibrewise (strong) Strøm structure $h|_{V_{m+1}} : V_{m+1} \times [0, 1] \rightarrow X$ gives a fibrewise pointed contraction of $V_{m+1} \supset \Delta(B)$. Since $V_i \cup V_{m+1} = U'_i \cup V_{m+1} = U_i \cup V_{m+1} \supset U_i$, we obtain $\bigcup_{i=0}^{m+1} V_i = \bigcup_{i=0}^m (V_i \cup V_{m+1}) \supset \bigcup_{i=0}^m U_i = X$. This implies $\text{cat}_B^B(X) \leq m + 1 = \text{cat}_B^*(X) + 1$ and it completes the proof of Theorem 1. \square

Theorem 2. *Let X be a fibrewise well-pointed space over B with $\text{cat}_B^*(X) = m$ and $\{U_i; 0 \leq i \leq m\}$ be an open cover of X , in which U_i is fibrewise contractible (into $s_X(B)$) by a fibrewise homotopy $H_i : U_i \times [0, 1] \rightarrow X$. Then we have $\text{cat}_B^B(X) = m = \text{cat}_B^*(X)$ if one of the following conditions is satisfied.*

- (1) *There exists i , $0 \leq i \leq m$ such that U_i does not intersect with $s_X(B)$.*
- (2) *There exists i , $0 \leq i \leq m$ such that U_i includes $s_X \circ p_X(U_i) \subset X$.*

Theorem 2 immediately implies the following corollary.

Corollary 3. *Let B be a locally finite simplicial complex with $\mathcal{TC}(B) = m$ and $\{U_i; 1 \leq i \leq m\}$ be an open cover of X , in which U_i is compressible into the image $\Delta(B)$ of diagonal map $\Delta : B \rightarrow B \times B$. Then we have $\mathcal{TC}^M(B) = m = \mathcal{TC}(B)$ if one of the following conditions is satisfied.*

- (1) *There exists i , $1 \leq i \leq m$ such that U_i does not intersect with $\Delta(B)$.*
- (2) *There exists i , $1 \leq i \leq m$ such that U_i includes $\Delta \circ \text{pr}_2(U_i) \subset B \times B$.*

Proof of Theorem 2: For simplicity, we assume that $i = 0$ in each cases. Let X be a fibrewise well-pointed space over B with a projection $p_X : X \rightarrow B$ and a section $s_X : B \rightarrow X$. Let (u, h) be a fibrewise (strong) Strøm structure on $(X, s_X(B))$, i.e., $u : X \rightarrow [0, 1]$ is a map and $h : X \times [0, 1] \rightarrow X$ is a fibrewise pointed homotopy such that $u^{-1}(0) = s_X(B)$, $h(x, 0) = x$ for any $x \in X$ and $h(x, 1) = s_X \circ p_X(x)$ for any $x \in X$ with $u(x) < 1$. Then the fibrewise map $r : X \rightarrow X$ given by $r(x) = h(x, 1)$ satisfies the following.

- i) $X = \bigcup_{i=0}^m r^{-1}(U_i)$, since $X = \bigcup_{i=0}^m U_i$.
- ii) r is fibrewise homotopic to the identity by h .
- iii) $r^{-1}(s_X(B)) \supset U = u^{-1}([0, 1))$, where U is fibrewise contractible by $h|_U$.

- iv) Each $r^{-1}(U_i)$ is fibrewise contractible, since r is fibrewise homotopic to the identity by ii) and U_i is fibrewise contractible.

Firstly, we consider the case (1): let $V_0 = r^{-1}(U_0) \cup u^{-1}([0, \frac{2}{3}))$ and $V_i = (r^{-1}(U_i) \setminus u^{-1}([0, \frac{1}{2}])) \cup u^{-1}([0, \frac{1}{3}))$, $1 \leq i \leq m$. Thus $\bigcup_{i=0}^m V_i = r^{-1}(U_0) \cup \bigcup_{i=1}^m (V_i \cup u^{-1}([0, \frac{2}{3}))) \supset r^{-1}(U_0) \cup \bigcup_{i=1}^m r^{-1}(U_i) = \bigcup_{i=0}^m r^{-1}(U_i) = X$ by i). Since $r^{-1}(U_i)$ is fibrewise contractible by iv), so is the open set $r^{-1}(U_i) \setminus u^{-1}([0, \frac{1}{2}])$ for every $i \geq 0$, where $r^{-1}(U_0) \setminus u^{-1}([0, \frac{1}{2}]) = r^{-1}(U_0)$ since U_0 does not intersect with $s_X(B)$. On the other hand, $u^{-1}([0, \frac{t}{3}))$, $t = 1, 2$ are also fibrewise contractible by fibrewise pointed homotopies by iii). Hence each V_i , $0 \leq i \leq m$ is fibrewise contractible by a fibrewise pointed homotopy, and hence $\text{cat}_B^B(X) \leq m = \text{cat}_B^*(X)$. Thus we have $\text{cat}_B^*(X) = \text{cat}_B^B(X)$.

Secondly, we consider the case (2): let $V_0 = r^{-1}(U_0) \cup u^{-1}([0, \frac{2}{3}))$ and $V_i = (r^{-1}(U_i) \setminus u^{-1}([0, \frac{1}{2}])) \cup u^{-1}([0, \frac{1}{3}))$, $1 \leq i \leq m$. Thus $\bigcup_{i=0}^m V_i = r^{-1}(U_0) \cup \bigcup_{i=1}^m (V_i \cup u^{-1}([0, \frac{2}{3}))) \supset \bigcup_{i=0}^m r^{-1}(U_i) = X$ by i). Since $r^{-1}(U_i)$ is fibrewise contractible by iv), so is the open set $r^{-1}(U_i) \setminus u^{-1}([0, \frac{1}{2}])$ which does not intersect with $u^{-1}([0, \frac{1}{3}))$, for every $i > 0$. On the other hand, each open set $u^{-1}([0, \frac{t}{3}))$, $t = 1, 2$ is fibrewise contractible by a fibrewise pointed homotopy by iii). Hence each open set V_i , $1 \leq i \leq m$ is fibrewise contractible by fibrewise pointed homotopy. When $i = 0$, we need to construct a fibrewise pointed homotopy $H : V_0 \times [0, 1] \rightarrow X$ using the fibrewise homotopy $H_0 : U_0 \times [0, 1] \rightarrow X$ and the fibrewise (strong) Strøm structure (u, h) as follows:

$$H(x, t) = \begin{cases} \left\{ \begin{array}{ll} x, & t = 0 \\ h(x, 3t), & 0 \leq t \leq \frac{1}{3} \\ r(x), & t = \frac{1}{3} \\ H_0(r(x), 3t-1), & \frac{1}{3} \leq t \leq \frac{2}{3} \\ s_X \circ p_X(r(x)) = s_X \circ p_X(x) = s_X(b), & t = \frac{2}{3} \\ H_0(s_X(b), 3-3t), & \frac{2}{3} \leq t \leq 1 \\ s_X(b), & t = 1 \end{array} \right\}, & x \in V_0 \setminus U, \\ \left\{ \begin{array}{ll} x, & t = 0 \\ h(x, 3t), & 0 \leq t \leq \frac{1}{3} \\ r(x) = s_X(b), & t = \frac{1}{3} \\ H_0(s_X(b), 3t-1), & \frac{1}{3} \leq t \leq u(x) - \frac{1}{3} \\ H_0(s_X(b), 3u(x)-2), & u(x) - \frac{1}{3} \leq t \leq \frac{5}{3} - u(x) \\ H_0(s_X(b), 3-3t), & \frac{5}{3} - u(x) \leq t \leq 1 \\ s_X(b), & t = 1 \end{array} \right\}, & \frac{2}{3} \leq u(x) < 1, \\ \left\{ \begin{array}{ll} x, & t = 0 \\ h(x, 3t), & 0 \leq t \leq \frac{1}{3} \\ r(x) = s_X(b), & \frac{1}{3} \leq t \leq 1 \end{array} \right\}, & 0 \leq u(x) < \frac{2}{3}, \\ s_X(b), & x \in s_X(B), \end{cases}$$

where $b = p_X(x) = p_X(r(x))$, and hence for $x \in V_0 \setminus u^{-1}([0, \frac{2}{3})) \subset r^{-1}(U_0)$, we have $s_X(b) = s_X \circ p_X(r(x)) \in U_0$ since $r(x) \in U_0$. Thus we have $\text{cat}_B^*(X) = \text{cat}_B^B(X)$, and it completes the proof of Theorem 2. \square

The following are corrections in [2].

- The part of Proof of Theorem 1.12 from page 13 line -3 to page 14 line 12 is not clearly given and must be rewritten completely:

Proof: For each vertex β of B , let V_β be the star neighbourhood in B and $V = \bigcup_\beta V_\beta \times V_\beta \subset B \times B$. Then the closure $\bar{V} = \bigcup_\beta \bar{V}_\beta \times \bar{V}_\beta$ is a subcomplex of $B \times B$. For the barycentric coordinates $\{\xi_\beta\}$ and $\{\eta_\beta\}$ of x and y , resp, we see that $(x, y) \in V$ if and only if $\sum_\beta \text{Min}(\xi_\beta, \eta_\beta) > 0$ and that $\sum_\beta \text{Min}(\xi_\beta, \eta_\beta) = 1$ if and only if the barycentric coordinates of x and y are the same, or equivalently, $(x, y) \in \Delta(B)$. Hence we can define a continuous map $v : B \times B \rightarrow [0, 3]$ by the following formula.

$$v(x, y) = \begin{cases} 3 - 3 \sum_\beta \text{Min}(\xi_\beta, \eta_\beta), & \text{if } (x, y) \in \bar{V}, \\ 3, & \text{if } (x, y) \notin \bar{V}. \end{cases}$$

Since B is locally finite, v is well-defined on $B \times B$, and we have $v^{-1}(0) = \Delta(B)$ and $v^{-1}([0, 3)) = V$. Let $U = v^{-1}([0, 1))$ an open neighbourhood of $\Delta(B)$. In [3], Milnor defined a map $\mu : V \rightarrow B$ giving an ‘average’ of $(x, y) \in V$ as follows.

$$\mu(x, y) = (\zeta_\beta), \quad \zeta_\beta = \text{Min}(\xi_\beta, \eta_\beta) / \sum_\gamma \text{Min}(\xi_\gamma, \eta_\gamma),$$

where $\{\xi_\beta\}$ and $\{\eta_\beta\}$ are barycentric coordinates of x and y respectively, and γ runs over all vertices in B . Since B is locally finite, μ is well-defined on V and satisfies $\mu(x, x) = x$ for any $x \in B$. Using the map μ , Milnor introduced a map $\lambda : V \times [0, 1] \rightarrow B$ as follows.

$$\lambda(x, y, t) = \begin{cases} (1-2t)x + 2t\mu(x, y), & t \leq \frac{1}{2}, \\ (2-2t)\mu(x, y) + (2t-1)y, & t \geq \frac{1}{2}. \end{cases}$$

Hence we have $\lambda(x, x, t) = x$ for any $x \in B$ and $t \in [0, 1]$. Using Milnor’s map λ , we obtain a pair of maps (u, h) as follows:

$$u(x, y) = \text{Min}\{1, v(x, y)\} \quad \text{and} \\ h(x, y, t) = \begin{cases} (\lambda(x, y, \text{Min}\{t, w(x, y)\}), y), & \text{if } v(x, y) < 3, \\ (x, y), & \text{if } v(x, y) > 2, \end{cases}$$

where $w : B \times B \rightarrow [0, 1]$ is given by

$$w(x, y) = \begin{cases} 1, & v(x, y) \leq 1, \\ 2 - v(x, y), & 1 \leq v(x, y) \leq 2, \\ 0, & v(x, y) \geq 2. \end{cases}$$

If $2 < v(x, y) < 3$, then, by definition, we have $w(x, y) = 0$ and

$$(\lambda(x, y, \text{Min}\{t, w(x, y)\}), y) = (\lambda(x, y, 0), y) = (x, y).$$

Thus h is also a well-defined continuous map. Then we have $u^{-1}(0) = \Delta(B)$, $u^{-1}([0, 1)) = U$ and $h(x, y, 0) = (x, y)$ for any $(x, y) \in B \times B$. If

$(x, y) \in U$, we have $w(x, y) = 1$, $h(x, y, t) = (\lambda(x, y, t), y)$ and $h(x, y, 1) = (y, y) \in \Delta(B)$. Moreover, we have $h(x, x, t) = (x, x)$ for any $x \in B$ and $t \in [0, 1]$ and $\text{pr}_2 \circ h(x, y, t) = y$ for any $(x, y, t) \in B \times B \times [0, 1]$. This implies that h is a fibrewise pointed homotopy. Thus the data (u, h) gives the fibrewise (strong) Strøm structure on $(B \times B, \Delta(B))$. \square

- In page 19, line 34, “ $t = 0$ ” must be replaced by “ $t = 1$ ”.
- In page 20, line 17, “that” must be replaced by “that $H(s_Z(b), t) = s_W(b)$ for any $b \in B$ and”.
- In page 20, line 28, the formula “ $\check{H}(q(s_Z(b), t), s) = s_W(b)$,” must be added.

REFERENCES

- [1] M. C. Crabb. and I. M. James, “Fibrewise Homotopy Theory”, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 1998.
- [2] N. Iwase and M. Sakai, *Topological complexity is a fibrewise L-S category*, Topology and its Applications, **157** (2010), 10–21.
- [3] J. Milnor, *On Spaces Having the Homotopy Type of a CW-Complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.

E-mail address: iwase@math.kyushu-u.ac.jp

E-mail address, Sakai: sakai@kurume-nct.ac.jp

(Iwase) FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, FUKUOKA 810-8560, JAPAN

(Sakai) KURUME NATIONAL COLLEGE OF TECHNOLOGY, FUKUOKA 830-8555, JAPAN.

TOPOLOGICAL COMPLEXITY IS A FIBREWISE L-S CATEGORY

NORIO IWASE AND MICHIIHIRO SAKAI

ABSTRACT. Topological complexity $\mathcal{TC}(B)$ of a space B is introduced by M. Farber to measure how much complex the space is, which is first considered on a configuration space of a motion planning of a robot arm. We also consider a stronger version $\mathcal{TC}^M(B)$ of topological complexity with an additional condition: in a robot motion planning, a motion must be stasis if the initial and the terminal states are the same. Our main goal is to show the equalities $\mathcal{TC}(B) = \text{cat}_B^*(d(B)) + 1$ and $\mathcal{TC}^M(B) = \text{cat}_B^B(d(B)) + 1$, where $d(B) = B \times B$ is a fibrewise pointed space over B whose projection and section are given by $p_{d(B)} = \text{pr}_2 : B \times B \rightarrow B$ the canonical projection to the second factor and $s_{d(B)} = \Delta_B : B \rightarrow B \times B$ the diagonal. In addition, our method in studying fibrewise L-S category is able to treat a fibrewise space with singular fibres.

1. INTRODUCTION

We say a pair of spaces (X, A) is an NDR pair or A is an NDR subset of X , if the inclusion map is a (closed) cofibration, in other words, the inclusion map has the (strong) Strøm structure (see page 22 in G. Whitehead [24]). When the set of the base point of a space is an NDR subset, the space is called well-pointed.

Let us recall the definition of a sectional category (see James [14]) which is originally defined and called by Schwarz ‘genus’.

Definition 1.1 (Schwarz [21], James [15]). *For a fibration $p : E \rightarrow X$, the sectional category $\text{secat}(p)$ (= one less than the Schwarz genus $\text{Genus}(p)$) is the minimal number $m \geq 0$ such that there exists a cover of X by $(m+1)$ open subsets $U_i \subset X$ each of which admits a continuous section $s_i : U_i \rightarrow E$.*

The topological complexity of a robot motion planning is first introduced by M. Farber [2] in 2003 to measure the discontinuity of a robot motion planning algorithm searching also the way to minimise the discontinuity. At a more general view point, Farber defined a numerical invariant $\mathcal{TC}(B)$ of any topological space B : let $\mathcal{P}(B)$ be the space of all paths in B . Then there is a Serre path fibration $\pi : \mathcal{P}(B) \rightarrow B \times B$ given by $\pi(\ell) = (\ell(0), \ell(1))$ for $\ell \in \mathcal{P}(B)$.

Definition 1.2 (Farber). *For a space B , the topological complexity $\mathcal{TC}(B)$ is the minimal number $m \geq 1$ such that there exists a cover of $B \times B$ by m open subsets U_i each of which admits a continuous section $s_i : U_i \rightarrow \mathcal{P}(B)$ for $\pi : \mathcal{P}(B) \rightarrow B \times B$.*

By definition, we can observe that the topological complexity is nothing but the Schwartz genus or the sectional category.

Date: February 28, 2012, [First draft].

2000 Mathematics Subject Classification. Primary 55M30, Secondary 55Q25.

Key words and phrases. Topological complexity, Lusternik-Schnirelmann category.

Farber has further introduced a new invariant restricting motions by giving two additional conditions on the section $s : U \rightarrow \mathcal{P}(B)$ (see Farber [3]).

- (1) $s(b, b) = c_b$ the constant path at b for any $b \in B$,
- (2) $s(b_1, b_2) = s(b_2, b_1)^{-1}$ if $(b_1, b_2) \in U$.

It gives a stronger invariant than the topological complexity, and the $\mathbb{Z}/2$ -equivariant theory must be applied as in Farber-Grant [4]. This new topological invariant, in turn, suggests us another motion planning under the condition that a motion is stasis if the initial and the terminal states are the same. Let us state more precisely.

Definition 1.3. For a space B , the ‘monoidal’ topological complexity $\mathcal{TC}^M(B)$ is the minimal number $m \geq 1$ such that there exists a cover of $B \times B$ by m open subsets $U_i \supset \Delta(B)$ each of which admits a continuous section $s_i : U_i \rightarrow \mathcal{P}(B)$ for the Serre path fibration $\pi : \mathcal{P}(B) \rightarrow B \times B$ satisfying $s_i(b, b) = c_b$ for any $b \in B$.

Remark 1.4. This new topological complexity \mathcal{TC}^M is **not** a homotopy invariant, in general. However, it is a homotopy invariant if we restrict our working category to the category of a space B such that the pair $(B \times B, \Delta(B))$ is NDR.

On the other hand, a fibrewise pointed L-S category of a fibrewise pointed space is introduced and studied by James-Morris [13]. Let us recall the definition:

Definition 1.5 (James-Morris [13]). (1) Let X be a fibrewise pointed space over B . The fibrewise **pointed** L-S category $\text{cat}_B^B(X)$ is the minimal number $m \geq 0$ such that there exists a cover of X by $(m+1)$ open subsets $U_i \supset s_X(B)$ each of which is fibrewise null-homotopic in X by a fibrewise pointed homotopy. If there are no such m , we say $\text{cat}_B^B(X) = \infty$.
 (2) Let $f : Y \rightarrow X$ be a fibrewise pointed map over B . The fibrewise **pointed** L-S category $\text{cat}_B^B(f)$ is the minimal number $m \geq 0$ such that there exists a cover of Y by $(m+1)$ open subsets $U_i \supset s_Y(B)$, where the restriction $f|_{U_i}$ to each subset is fibrewise compressible into $s_X(B)$ in X by a fibrewise pointed homotopy. If there are no such m , we say $\text{cat}_B^B(f) = \infty$.

To describe our main result, we further introduce a new unpointed version of fibrewise L-S category: the fibrewise L-S category $\text{cat}_B(\)$ of an fibrewise *unpointed* space is also defined by James and Morris [13] as the minimum number (minus one) of open subsets which cover the given space and are fibrewise null-homotopic (see also James [14] and Crabb-James [1]). In this paper, we give a new version of a fibrewise *unpointed* L-S category of a fibrewise *pointed* space as follows:

Definition 1.6. (1) Let X be a fibrewise pointed space over B . The fibrewise **unpointed** L-S category $\text{cat}_B^*(X)$ is the minimal number $m \geq 0$ such that there exists a cover of X by $(m+1)$ open subsets U_i each of which is fibrewise compressible into $s_X(B)$ in X by a fibrewise homotopy. If there are no such m , we say $\text{cat}_B^*(X) = \infty$.
 (2) Let $f : Y \rightarrow X$ be a fibrewise pointed map over B . The fibrewise **unpointed** L-S category $\text{cat}_B^*(f)$ is the minimal number $m \geq 0$ such that there exists a cover of Y by $(m+1)$ open subsets U_i , where the restriction $f|_{U_i}$ to each subset is fibrewise compressible into $s_X(B)$ in X by a fibrewise homotopy. If there are no such m , we say $\text{cat}_B^*(f) = \infty$.

For a given space B , we define a fibrewise pointed space $d(B)$ by $d(B) = B \times B$ with $p_{d(B)} = \text{pr}_2 : B \times B \rightarrow B$ and $s_{d(B)} = \Delta_B : B \rightarrow B \times B$ the diagonal. One of our main goals of this paper is to show the following theorem.

Theorem 1.7. *For a space B , we have the following equalities.*

- (1) $\mathcal{TC}(B) = \text{cat}_B^*(d(B)) + 1.$
- (2) $\mathcal{TC}^M(B) = \text{cat}_B^B(d(B)) + 1.$

Farber and Grant has also introduced lower bounds for the topological complexity by using the cup length and category weight (see Rudyak [17, 18] for example) on the ideal of zero-divisors, i.e, the kernel of $\Delta^* : H^*(B \times B; R) \rightarrow H^*(B; R).$

Definition 1.8 (Farber [2] and Farber-Grant [4]). *For a space B and a ring $R \ni 1$, the zero-divisors cup-length $\mathcal{Z}_R(B)$ and the TC-weight $\text{wgt}_\pi(u; R)$ for $u \in I = \ker \Delta^* : H^*(B \times B; R) \rightarrow H^*(B; R)$ is defined as follows.*

- (1) $\mathcal{Z}_R(B) = \text{Max} \{m \geq 0 \mid H^*(B \times B; R) \supset I^m \neq 0\}$
- (2) $\text{wgt}_\pi(u; R) = \text{Max} \{m \geq 0 \mid \forall f : Y \rightarrow B \times B \text{ (secat}(f^*\pi) < m), f^*(u) = 0\}$

In the category $\underline{\mathcal{T}}_B^B$ of fibrewise pointed spaces with base space B and maps between them, we also have corresponding definitions.

Definition 1.9. *For a fibrewise pointed space X over B and a ring $R \ni 1$ and $u \in I = H^*(X, B; R) \subset H^*(X; R)$, we define*

- (1) $\text{cup}_B^B(X; R) = \text{Max} \{m \geq 0 \mid \exists \{u_1, \dots, u_m \in H^*(X, B; R)\} \text{ s.t. } u_1 \cdots u_m \neq 0\}$
- (2) $\text{wgt}_B^B(u; R) = \text{Max} \left\{ m \geq 0 \mid \forall f : Y \rightarrow X \in \underline{\mathcal{T}}_B^B \text{ (cat}_B^B(f) < m), f^*(u) = 0 \right\}$

This immediately implies the following.

Theorem 1.10. *For a space B , we have $\mathcal{Z}_R(B) = \text{cup}_B^B(d(B); R)$ for a ring $R \ni 1$.*

Motivating by this equality, we proceed to obtain the following result.

Theorem 1.11. *For any space B , any element $u \in H^*(B \times B, \Delta(B); R)$ and a ring $R \ni 1$, we have $\text{wgt}_\pi(u; R) = \text{wgt}_B^B(u; R).$*

Let us consider one technical condition on a fibrewise pointed space:

Theorem 1.12. *For any space B having the homotopy type of a locally finite simplicial complex, we may assume that $d(B)$ is fibrewise well-pointed up to homotopy.*

The following is the main result of our paper.

Theorem 1.13. *For any fibrewise well-pointed space X over B , we have $\text{cat}_B^B(X) = \text{cat}_B^*(X)$. So, if B is a locally finite simplicial complex, we have $\mathcal{TC}(B) = \mathcal{TC}^M(B) = \text{cat}_B^B(d(B)) + 1.$*

In [19], Sakai showed, in his study of the fibrewise *pointed* L-S category of a fibrewise well-pointed spaces, using Whitehead style definition, that we can utilise A_∞ methods used in the study of L-S category (see Iwase [7, 8]). Let us state the Whitehead style definitions of fibrewise L-S categories following [19].

Definition 1.14. *Let X be a fibrewise well-pointed space over B . The fibrewise **pointed** L-S category $\text{cat}_B^B(X)$ is the minimal number $m \geq 0$ such that the $(m+1)$ -fold fibrewise diagonal $\Delta_B^{m+1} : X \rightarrow \prod_B^{m+1} X$ is compressible into the fibrewise fat wedge $\mathbb{T}_B^{m+1} X$ in $\underline{\mathcal{T}}_B^B$. If there are no such m , we say $\text{cat}_B^B(X) = \infty$.*

We remark that this new definition coincides with the ordinary one, if the total space X is a finite simplicial complex.

The above Whitehead-style definition allows us to define the module weight, cone length and categorical length, and moreover, to give their relationship as in Section 8. To show that, we need a criterion given by fibrewise A_∞ structure on the fibrewise loop space (see Sections 6–7).

2. PROOF OF THEOREM 1.7

First, we show the equality $\mathcal{TC}^M(B) = \text{cat}_B^B(d(B)) + 1$: assume $\mathcal{TC}^M(B) = m+1$, $m \geq 0$ and that there are an open cover $\bigcup_{i=0}^m U_i = B \times B$ and a series of sections $s_i : U_i \rightarrow \mathcal{P}(B)$ of $\pi : \mathcal{P}(B) \rightarrow d(B)$ satisfying $s_i(b, b) = c_b$ for $b \in B$, since we are considering monoidal topological complexity. Then each U_i is fibrewise compressible relative to $\Delta(B)$ into $\Delta(B) \subset B \times B = d(B)$ by a homotopy $H_i : U_i \times [0, 1] \rightarrow B \times B$ given by the following:

$$H_i(a, b; t) = (s_i(a, b)(t), b), \quad (a, b) \in U_i, \quad t \in [0, 1],$$

where we can easily check that H_i gives a fibrewise compression of U_i relative to $\Delta(B)$ into $\Delta(B) \subset B \times B$. Since $\bigcup_{i=0}^m U_i = B \times B = d(B)$, we obtain $\text{cat}_B^B(d(B)) \leq m$, and hence we have $\text{cat}_B^B(d(B)) + 1 \leq \mathcal{TC}^M(B)$.

Conversely assume that $\text{cat}_B^B(d(B)) = m$, $m \geq 0$ and there is an open cover $\bigcup_{i=0}^m U_i = d(B)$ of $d(B) = B \times B$ where U_i is fibrewise compressible relative to $\Delta(B)$ into $\Delta(B) \subset d(B) = B \times B$: let us denote the compression homotopy of U_i by $H_i(a, b; t) = (\sigma_i(a, b; t), b)$ for $(a, b) \in U_i$ and $t \in [0, 1]$, where $\sigma_i(a, b; 0) = a$ and $\sigma_i(a, b; 1) = b$. Hence we can define a section $s_i : U_i \rightarrow \mathcal{P}(B)$ by the formula

$$s_i(a, b)(t) = \sigma_i(a, b; t) \quad t \in [0, 1].$$

Since $\bigcup_{i=0}^m U_i = B \times B$, we obtain $\mathcal{TC}^M(B) \leq m+1$ and hence we have $\mathcal{TC}^M(B) \leq \text{cat}_B^B(d(B)) + 1$. Thus we have $\mathcal{TC}^M(B) = \text{cat}_B^B(d(B)) + 1$.

Second, we show the equality $\mathcal{TC}(B) = \text{cat}_B^*(d(B)) + 1$: assume $\mathcal{TC}(B) = m+1$, $m \geq 0$ and that there is a open cover $\bigcup_{i=0}^m U_i = B \times B$ and a section $s_i : U_i \rightarrow \mathcal{P}(B)$ of $\pi : \mathcal{P}(B) \rightarrow d(B)$. Then each U_i is fibrewise compressible into $\Delta(B) \subset B \times B = d(B)$ by a homotopy $H_i : U_i \times [0, 1] \rightarrow B \times B$ which is given by

$$H_i(a, b; t) = (s_i(a, b)(t), b), \quad (a, b) \in U_i, \quad t \in [0, 1],$$

where we can easily check that H gives a fibrewise compression of U_i into $\Delta(B) \subset B \times B = d(B)$. Since $\bigcup_{i=0}^m U_i = B \times B = d(B)$, we obtain $\text{cat}_B^*(d(B)) \leq m$, and hence we have $\text{cat}_B^*(d(B)) + 1 \leq \mathcal{TC}(B)$.

Conversely assume that $\text{cat}_B^*(d(B)) = m$, $m \geq 0$ and there is an open cover $\bigcup_{i=0}^m U_i = d(B)$ of $d(B) = B \times B$ where U_i is fibrewise compressible into $\Delta(B) \subset B \times B = d(B)$: the compression homotopy is described as $H_i(a, b; t) = (\sigma_i(a, b; t), b)$ for $(a, b) \in U_i$ and $t \in [0, 1]$, such that $\sigma_i(a, b; 0) = a$ and $\sigma_i(a, b; 1) = b$. Hence we can define a section $s_i : U_i \rightarrow \mathcal{P}(B)$ by the formula

$$s_i(a, b)(t) = \sigma_i(a, b; t) \quad t \in [0, 1].$$

Since $\bigcup_{i=0}^m U_i = B \times B$, we obtain $\mathcal{TC}(B) \leq m+1$ and hence we have $\mathcal{TC}(B) \leq \text{cat}_B^*(d(B)) + 1$. Thus we have $\mathcal{TC}(B) = \text{cat}_B^*(d(B)) + 1$. \square

3. PROOF OF THEOREM 1.11

Assume that $\text{wgt}_B^B(u; R) = m$, where $u \in H^*(B \times B, \Delta(B))$ and $f : Y \rightarrow d(B) = B \times B$ a map of $\text{secat}(f^* \pi) < m$. Then there is an open cover $\bigcup_{i=1}^m U_i = Y$ and a series of maps $\{\sigma_i : U_i \rightarrow \mathcal{P}(B); 1 \leq i \leq m\}$ satisfying $\pi \circ \sigma_i = f|_{U_i}$. Let $\hat{Y} = Y \amalg B$ with projection $p_{\hat{Y}}$ and section $s_{\hat{Y}}$ given by

$$p_{\hat{Y}}|_Y = p_Y, \quad p_{\hat{Y}}|_B = \text{id}_B \quad \text{and} \quad s_{\hat{Y}} : B \hookrightarrow Y \amalg B = \hat{Y}.$$

Then we can extend f to a map $\hat{f} : \hat{Y} \rightarrow d(B)$ by the formula

$$\hat{f}|_Y = f, \quad \hat{f}|_B = s_{d(B)} = \Delta.$$

By putting $\hat{U}_i = U_i \amalg B$ which is open in \hat{Y} , we obtain an open cover $\bigcup_{i=1}^m \hat{U}_i = \hat{Y}$ and a series of maps $\hat{\sigma}_i : \hat{U}_i \rightarrow \mathcal{P}(B)$ satisfying $\pi \circ \hat{\sigma}_i = \hat{f}|_{\hat{U}_i}$:

$$\hat{\sigma}_i|_{U_i} = \sigma_i, \quad \hat{\sigma}_i|_B = s_{\mathcal{P}(B)}.$$

Hence there is a fibrewise homotopy $\Phi_i : \hat{U}_i \times [0, 1] \rightarrow d(B)$ such that $\Phi_i(y, 0) = \hat{f}(y)$ and $\Phi_i(y, 1) \in \Delta(B)$ given by the following formula.

$$\Phi_i(y, t) = (\hat{\sigma}_i(y)(t), \hat{\sigma}_i(y)(1)), \quad (y, t) \in \hat{U}_i \times [0, 1],$$

so that we have $\Phi_i(y, 0) = (\hat{\sigma}_i(y)(0), \hat{\sigma}_i(y)(1)) = \pi \circ \hat{\sigma}_i(y) = \hat{f}(y)$ and $\Phi_i(y, 1) = (\hat{\sigma}_i(y)(1), \hat{\sigma}_i(y)(1)) \in \Delta(B)$. Moreover, for any $(b, t) \in B \times [0, 1]$, we have $\Phi_i(b, t) = (\hat{\sigma}_i(b)(t), \hat{\sigma}_i(b)(1)) = (s_{\mathcal{P}(B)}(t), s_{\mathcal{P}(B)}(1)) = (b, b)$. Thus Φ_i gives a fibrewise pointed compression homotopy of $\hat{f}|_{\hat{U}_i}$ into $\Delta(B)$. Then it follows that $\text{cat}_B^B(\hat{f}) < m$ and hence we obtain $f^*(u) = 0$ and $\text{wgt}_\pi(u; R) \geq m$. Thus we obtain $\text{wgt}_\pi(u; R) \geq m = \text{wgt}_B^B(u; R)$.

Conversely assume that $\text{wgt}_\pi(u; R) = m$, where $u \in H^*(B \times B, \Delta(B))$ and $f : Y \rightarrow B \times B$ such that $\text{cat}_B^B(f) < m$. Then there exists an open covering $\bigcup_{i=1}^m U_i = Y$ with $U_i \supset s_Y(B)$ and a sequence of fibrewise homotopies $\{\phi_i : U_i \times [0, 1] \rightarrow B \times B\}$ such that $\phi_i(y, 0) = f|_{U_i}(y)$, $\phi_i(y, 1) \in \Delta(B)$ and $\text{pr}_2 \circ \phi_i(y, t) = \text{pr}_2 \circ f(y)$ for $(y, t) \in U_i \times [0, 1]$. Hence there is a sequence of maps $\{\sigma_i : U_i \rightarrow \mathcal{P}(B)\}$ given by

$$\sigma_i(y)(t) = \text{pr}_1 \circ \phi_i(y, t), \quad y \in U_i, \quad t \in [0, 1]$$

such that $\pi \circ \sigma_i(y) = (\text{pr}_1 \circ \phi_i(y, 0), \text{pr}_1 \circ \phi_i(y, 1)) = f(y)$ since $\text{pr}_2 \circ \phi_i(y, t) = \text{pr}_2 \circ f(y)$ for $(y, t) \in U_i \times [0, 1]$. Thus we obtain $\text{secat}(f^* \pi) < m$, and hence $f^*(u) = 0$. This implies $\text{wgt}_B^B(u; R) \geq m = \text{wgt}_\pi(u; R)$ and hence $\text{wgt}_B^B(u; R) = \text{wgt}_\pi(u; R)$. \square

4. PROOF OF THEOREM 1.12

The proof of Lemma 2 in §2 of Milnor [16] implies the following:

Lemma 4.1. *The pair $(B \times B, \Delta(B))$ is an NDR-pair.*

Proof: For each vertex β of B , let V_β be the star neighbourhood in B and $V = \bigcup_\beta V_\beta \times V_\beta \subset B \times B$. Then the closure $\bar{V} = \bigcup_\beta \bar{V}_\beta \times \bar{V}_\beta$ is a subcomplex of $B \times B$. For the barycentric coordinates $\{\xi_\beta\}$ and $\{\eta_\beta\}$ of x and y , resp, we see that $(x, y) \in V$ if and only if $\sum_\beta \text{Min}(\xi_\beta, \eta_\beta) > 0$ and that $\sum_\beta \text{Min}(\xi_\beta, \eta_\beta) = 1$ if and only if the barycentric coordinates of x and y are the same, or equivalently, $(x, y) \in \Delta(B)$. Hence we can define a continuous map $v : B \times B \rightarrow [-1, 1]$ by the following formula.

$$v(x, y) = \begin{cases} 2 \sum_\beta \text{Min}(\xi_\beta, \eta_\beta) - 1, & \text{if } (x, y) \in \bar{V}, \\ -1, & \text{if } (x, y) \notin \bar{V}. \end{cases}$$

Then we have that $v^{-1}(1) = \Delta(B)$. Let $U = v^{-1}((0, 1])$ an open neighbourhood of $\Delta(B)$. Using Milnor's map s , we obtain a pair of maps (u, h) as follows:

$$\begin{aligned} u(x, y) &= \text{Min}\{1, 1-v(x, y)\} \quad \text{and} \\ h(x, y, t) &= (s(x, y)(\text{Min}\{t, w(x, y)\}), y), \end{aligned}$$

where $w(x, y) = u(x, y) + v(x, y) = \text{Min}\{1, 1+v(x, y)\}$. Note that $w(x, y) = 1$ if $(x, y) \in U$ and that $w(x, y) = 0$ if $(x, y) \notin U$. Then $u^{-1}(0) = \Delta(B)$, $u^{-1}([0, 1]) = U$ and $h(x, y, 1) = (y, y) \in \Delta(B)$ if $(x, y) \in U$. Moreover, $\text{pr}_2 \circ h(x, y, t) = y$ and $h(x, x, t) = (s(x, x)(t), x) = (x, x)$ for any $x, y \in B$ and $t \in [0, 1]$. Thus the data (u, h) gives the fibrewise Strøm structure on $(B \times B, \Delta(B))$. \square

5. PROOF OF THEOREM 1.13

Let X be a fibrewise well-pointed space over B and \hat{X} the fiberwise pointed space obtained from X by giving a fibrewise whisker. More precisely, we define \hat{X} be the mapping cylinder of s_X ,

$$\hat{X} = X \cup_{s_X} B \times [0, 1], \quad X \ni s_X(b) \sim (b, 0) \in B \times [0, 1] \text{ for any } b \in B,$$

with projection $p_{\hat{X}}$ and section $s_{\hat{X}}$ given by the formulas

$$\begin{aligned} p_{\hat{X}}|_X &= p_X, \quad p_{\hat{X}}|_{B \times [0, 1]}(b, t) = b, \quad \text{for } (b, t) \in B \times [0, 1], \\ s_{\hat{X}}(b) &= (b, 1) \in B \times [0, 1] \subset \hat{X}. \end{aligned}$$

Then by the definition of Strøm structure, X is fibrewise pointed homotopy equivalent to \hat{X} the fibrewise whiskered space over B . So we have $\text{cat}_B^B(X) = \text{cat}_B^B(\hat{X})$ and $\text{cat}_B^*(X) = \text{cat}_B^*(\hat{X})$.

Assume that $\text{cat}_B^B(X) = m \geq 0$. Then it is clear by definition that $\text{cat}_B^*(X) \leq m = \text{cat}_B^B(X)$.

Conversely assume that $\text{cat}_B^*(X) = m \geq 0$. Then there is an open cover $\bigcup_{i=0}^m U_i = X$ such that U_i is compressible into $s_X(B) \subset X$. Hence there is a fibrewise homotopy $\Phi_i : U_i \times [0, 1] \rightarrow X$ such that $\Phi_i(x, 0) = x$, $\Phi_i(x, 1) = s_X(p_X(x))$ and $p_X \circ \Phi_i(x, t) = p_X(x)$. We define \hat{U}_i as follows:

$$\hat{U}_i = U_i \cup_{s_X} (s_X)^{-1}(U_i) \times [0, 1] \cup B \times (\frac{2}{3}, 1].$$

We also define a fibrewise pointed homotopy $\hat{\Phi}_i : \hat{U}_i \times [0, 1] \rightarrow \hat{X}$ as follows:

$$\hat{\Phi}_i(\hat{x}, t) = \begin{cases} \Phi_i(x, t), & \hat{x} = x \in X, \\ \Phi_i(s_X(b), t-3s), & \hat{x} = (b, s) \in (s_X)^{-1}(U_i) \times (0, \frac{t}{3}), \\ s_X(b), & \hat{x} = (b, \frac{t}{3}), b \in (s_X)^{-1}(U_i), \\ (b, \frac{6s-2t}{6-3t}), & \hat{x} = (b, s) \in (s_X)^{-1}(U_i) \times (\frac{t}{3}, \frac{2}{3}), \\ (b, \frac{2}{3}), & \hat{x} = (b, \frac{2}{3}), b \in (s_X)^{-1}(U_i), \\ (b, s), & \hat{x} = (b, s) \in B \times (\frac{2}{3}, 1]. \end{cases}$$

It is then easy to see that \hat{U}_i 's cover the entire X , and hence we have $\text{cat}_B^B(\hat{X}) \leq m = \text{cat}_B^*(X)$. Thus $\text{cat}_B^B(X) \leq \text{cat}_B^*(X)$ and hence $\text{cat}_B^B(X) = \text{cat}_B^*(X)$. In particular, we have $\mathcal{TC}(B) = \mathcal{TC}^M(B)$ for a locally finite simplicial complex B . \square

6. FIBREWISE A_∞ STRUCTURES

From now on, we work in the category $\underline{\mathcal{T}}_B^B$. For any X a fibrewise pointed space over B , we denote by $p_X : X \rightarrow B$ its projection and by $s_X : B \rightarrow X$ its section.

We say that a pair (X, A) of fibrewise pointed spaces over B is a fibrewise NDR-pair or that A is a fibrewise NDR subset of X , if the inclusion map $A \hookrightarrow X$ is a fibrewise cofibration, in other words, the inclusion has the fibrewise (strong) Strøm structure (see Crabb-James [1]). Since B is the zero object in $\underline{\mathcal{T}}_B^B$, for any given fibrewise pointed space X over B , we always have a pair (X, B) in $\underline{\mathcal{T}}_B^B$, where we regard $s_X(B) = B$. When the pair (X, B) is fibrewise NDR, the space X is called fibrewise well-pointed.

Proposition 6.1 (Crabb-James [1]). (1) *If (X, A) and (X', A') are fibrewise NDR-pairs, then so is $(X, A) \times_B (X', A') = (X \times_B X', X \times_B A' \cup A \times_B X')$.*

(2) *If (X, A) is a fibrewise NDR-pair, then so is $(\prod_B^m X, \mathbb{T}_B^m(X, A))$, which is defined by induction for all $m \geq 1$:*

$$\begin{aligned} (\prod_B^1 X, \mathbb{T}_B^1(X, A)) &= (X, A), \\ (\prod_B^{m+1} X, \mathbb{T}_B^{m+1}(X, A)) &= (\prod_B^m X, \mathbb{T}_B^m(X, A)) \times_B (X, A). \end{aligned}$$

If X is a fibrewise pointed space over B , then by taking $A = B$, we obtain a fibrewise subspace $\mathbb{T}_B^{m+1}(X, B)$ of $\prod_B^{m+1} X$, which is called an $(m+1)$ -fold fibrewise fat-wedge of X , and is often denoted by $\mathbb{T}_B^{m+1} X$. In addition, the pair $(\prod_B^{m+1} X, \mathbb{T}_B^{m+1} X)$ is a fibrewise NDR-pair for all $m \geq 0$, if X is fibrewise well-pointed.

Examples 6.2. (1) *Let X be a fibrewise pointed space over B with $p_X = pr_2 : X = F \times B \rightarrow B$ the canonical projection to the second factor and $s_X = in_2 : B \hookrightarrow F \times B = E$ the canonical inclusion to the second factor. Then X is a fibrewise pointed space over B .*

(2) *Let $X = B \times B$, $p_X = pr_2 : B \times B \rightarrow B$ the canonical projection to the second factor and $s_X = \Delta_B : B \hookrightarrow B \times B$ the diagonal. Then X is a fibrewise pointed space over B .*

(3) *Let G be a topological group, EG the infinite join of G with right G action and $BG = EG/G$ the classifying space of G . By considering G as a left G space by the adjoint action, we obtain a fibrewise pointed space $X = EG \times_G G$ with $p_X : EG \times_G G \rightarrow BG$ with section $s_X : BG \hookrightarrow EG \times_G G \subseteq EG \times_G G$.*

(4) *Let B be a space, $X = \mathcal{L}(B)$ the space of free loops on B . Then $p_X : \mathcal{L}(B) \rightarrow B$ the evaluation map at $1 \in S^1 \subset \mathbb{C}$ is a fibration with section $s_X : B \rightarrow \mathcal{L}(B)$ given by the inclusion of constant loops. In view of Milnor's arguments, this example is homotopically equivalent to the example (3).*

Definition 6.3. *Let $\mathcal{P}_B(X) = \{\ell : [0, 1] \rightarrow X \mid \exists b \in B \text{ s.t. } \forall t \in [0, 1] \ p_X(\ell(t)) = b\}$ the fibrewise free path space, $\mathcal{L}_B(X) = \{\ell \in \mathcal{P}_B(X) \mid \ell(1) = \ell(0)\}$ the fibrewise free loop space and $\mathcal{L}_B^B(X) = \{\ell \in \mathcal{P}_B(X) \mid \ell(1) = \ell(0) = s_X \circ p_X(\ell(0))\}$ the fibrewise pointed loop space. For any $m \geq 0$, we define an A_∞ structure of $\mathcal{L}_B^B(X)$ as follows.*

- (1) $E_B^{m+1}(\mathcal{L}_B^B(X))$ as the homotopy pull-back in $\underline{\mathcal{T}}_B^B$ of $B \hookrightarrow \prod_B^{m+1} X \hookleftarrow \mathbb{T}_B^{m+1} X$,
- (2) $P_B^m(\mathcal{L}_B^B(X))$ as the homotopy pull-back in $\underline{\mathcal{T}}_B^B$ of $X \xrightarrow{\Delta_B^{m+1}} \prod_B^{m+1} X \hookleftarrow \mathbb{T}_B^{m+1} X$,

- (3) $e_m^X : P_B^m(\mathcal{L}_B^B(X)) \rightarrow X$ as the induced map from the inclusion $\overset{m+1}{T}_B X \hookrightarrow \overset{m+1}{\Pi}_B X$ by the diagonal $\Delta_B^{m+1} : X \rightarrow \overset{m+1}{\Pi}_B X$ and
- (4) $p_B^{\mathcal{L}_B^B(X)} : E_B^{m+1}(\mathcal{L}_B^B(X)) \rightarrow P_B^m(\mathcal{L}_B^B(X))$ as a map of fibrewise pointed spaces induced from the section $s_X : B \rightarrow X$, since the section $B \hookrightarrow \overset{m+1}{\Pi}_B X$ is nothing but the composition $\Delta_B^{m+1} \circ s_X : B \xrightarrow{s} X \xrightarrow{\Delta_B^{m+1}} \overset{m+1}{\Pi}_B X$.

We further investigate to understand an A_∞ stucture in a fiberwise view point, using fibrewise constructions. Clearly, these constructions are *not* exactly the Ganea-type fibre-cofibre constructions but the following.

Proposition 6.4 (Sakai). *Let X be a fibrewise pointed space over B and $m \geq 0$. Then $P_B^{m+1}(\mathcal{L}_B^B(X))$ has the homotopy type of a push-out of $p_B^{\mathcal{L}_B^B(X)} : E_B^{m+1}(\mathcal{L}_B^B(X)) \rightarrow P_B^m(\mathcal{L}_B^B(X))$ and the projection $E_B^{m+1}(\mathcal{L}_B^B(X)) \rightarrow B$.*

This is a direct consequence of the following lemma.

Lemma 6.5. *Let (X, A) and (X', A') be fibrewise NDR-pairs of fibrewise pointed spaces over B and Z a fibrewise pointed space over B with fibrewise maps $f : Z \rightarrow X$ and $g : Z \rightarrow X'$. Then the homotopy pull-back $\Omega_{(f,g),k}$ of maps $(f, g) : Z \rightarrow X \times_B X'$ and $k : X \times_B A' \cup A \times_B X' \hookrightarrow X \times_B X'$ has naturally the homotopy type of the reduced homotopy push-out $W = \Omega_{g,j} \cup_{p_2} \{ \Omega_{(f,g),i \times j} \wedge_B (B \times J^+) \} \cup_{p_1} \Omega_{f,i}$ of $p_1 : \Omega_{(f,g),i \times j} \rightarrow \Omega_{f,i}$ and $p_2 : \Omega_{(f,g),i \times j} \rightarrow \Omega_{g,j}$, where $J = [-1, 1]$ and*

$$\Omega_{(f,g),k} = \left\{ (z, \ell, \ell') \in Z \times_B \mathcal{P}_B(X) \times_B \mathcal{P}_B(X') \mid \begin{array}{l} f(z) = \ell(0), \ g(z) = \ell'(0), \\ (\ell(1), \ell'(1)) \in A \times_B X' \cup X \times_B A' \end{array} \right\},$$

$$\Omega_{(f,g),i \times j} = \{ (z, \ell, \ell') \in \Omega_{(f,g),k} \mid (\ell(1), \ell'(1)) \in A \times_B A' \},$$

$$\Omega_{f,i} = \{ (z, \ell) \in Z \times_B \mathcal{P}_B(X) \mid f(z) = \ell(0), \ \ell(1) \in A \},$$

$$\Omega_{g,j} = \{ (z, \ell') \in Z \times_B \mathcal{P}_B(X') \mid g(z) = \ell'(0), \ \ell'(1) \in A' \},$$

$$p_1(z, \ell, \ell') = (z, \ell) \text{ and } p_2(z, \ell, \ell') = (z, \ell').$$

Proof of Outline of the proof. The proof of Lemma 6.5 is quite similar to that of Theorem 1.1 in Sakai [20] (which is based on Iwase [7]) by replacing (Y, B) in [20] by (X', A') , defining and using the following spaces.

$$\widehat{W} = \Omega_{(f,g),i \times \text{id}_{X'}} \times \{-1\} \cup \{ \Omega_{(f,g),i \times j} \times J \} \cup \Omega_{(f,g),\text{id}_X \times j} \times \{1\} \subset \Omega_{(f,g),k} \times J,$$

$$\Omega_{(f,g),\text{id}_X \times j} = \{ (z, \ell, \ell') \in \Omega_{(f,g),k} \mid (\ell(1), \ell'(1)) \in X \times_B A' \},$$

$$\Omega_{(f,g),i \times \text{id}_{X'}} = \{ (z, \ell, \ell') \in \Omega_{(f,g),k} \mid (\ell(1), \ell'(1)) \in A \times_B X' \}.$$

The precise construction of homotopy equivalences and homotopies is identical to that in [20] and is left to the readers. \square

Theorem 6.6. *Let X be a fibrewise well-pointed space over B . Then the sequence $\{ p_B^{\mathcal{L}_B^B(X)} : E_B^{m+1}(\mathcal{L}_B^B(X)) \rightarrow P_B^m(\mathcal{L}_B^B(X)) \}$ gives a fibrewise pointed version of A_∞ -structure on the fibrewise pointed loop space $\mathcal{L}_B^B(X)$.*

Thus in the case when X is a fibrewise well-pointed space over B , we assume that $P_B^m(\mathcal{L}_B^B(X))$ is an increasing sequence given by homotopy push-outs with a fibrewise fibration $e_m^X : P_B^m(\mathcal{L}_B^B(X)) \rightarrow X$ such that $e_1^X : \mathcal{S}_B^B(\mathcal{L}_B^B(X)) \rightarrow X$ is a fibrewise evaluation.

- Examples 6.7.** (1) Let X be a fibrewise pointed space over B with $p_X = pr_2 : F \times B \rightarrow B$ the canonical projection and $s_X = in_2 : B \hookrightarrow F \times B$ the canonical inclusion. Then $\mathcal{L}_B^B(X) = \mathcal{L}(F) \times B$ is given by $p_{\mathcal{L}_B^B(X)} = pr_2 : \mathcal{L}(F) \times B \rightarrow B$ and $s_{\mathcal{L}_B^B(X)} = in_2 : B \hookrightarrow \mathcal{L}(F) \times B$.
- (2) Let $X = B \times B$ be a fibrewise pointed space over B with $p_X = pr_2 : B \times B \rightarrow B$ and $s_X = \Delta_B : B \hookrightarrow B \times B$ the diagonal. Then $\mathcal{L}_B^B(X) = \mathcal{L}(B)$ the free loop space on B , $p_{\mathcal{L}_B^B(X)} : \mathcal{L}(B) \rightarrow B$ the evalation map at $1 \in S^1 \subset \mathbb{C}$ and $s_{\mathcal{L}_B^B(X)} : B \hookrightarrow \mathcal{L}(B)$ the inclusion of constant loops.

Remark 6.8. When E is a cell-wise trivial fibration on a polyhedron B (see [12]), we can see that the canonical map $e_\infty^E : P_B^\infty(\mathcal{L}_B^B(E)) \rightarrow E$ is a homotopy equivalence by a similar arguments given in the proof of Theorem 2.9 of [12].

7. FIBREWISE L-S CATEGORIES OF FIBREWISE POINTED SPACES

The fibrewise *pointed* L-S category of an fibrewise pointed space is first defined by James and Morris [13] as the least number (minus one) of open subsets which cover the given space and are contractible by a homotopy fixing the base point in each fibre (see also James [14] and Crabb-James [1]) and is redefined by Sakai in [19] as follows:

let X be a fibrewise pointed space over B . For given $k \geq 0$, we denote by $\Pi_B^{k+1} X$ the $(k+1)$ -fold fibrewise product and by $T_B^{k+1} X$ the $(k+1)$ -fold fibrewise fat wedge. Then $\text{cat}_B^B(X) \leq m$ if the $(m+1)$ -fold fibrewise diagonal map $\Delta_B^{m+1} : X \rightarrow \Pi_B^{m+1} X$ is compressible into the fibrewise fat wedge $T_B^{m+1} X$ in $\underline{\mathcal{T}}_B^B$. If there is no such m , we say $\text{cat}_B^B(X) = \infty$. Let us consider the case when $\text{cat}_B^B(X) < \infty$. The definition of a fibrewise A_∞ structure yields the following criterion.

Theorem 7.1. Let X be a fibrewise pointed space over B and $m \geq 0$. Then $\text{cat}_B^B(X) \leq m$ if and only if $\text{id}_X : X \rightarrow X$ has a lift to $P_B^m(\mathcal{L}_B^B(X)) \xrightarrow{e_m^X} X$ in $\underline{\mathcal{T}}_B^B$.

Proof: If $\text{cat}_B^B(X) \leq m$, then the fibrewise diagonal $\Delta_B^{m+1} : X \rightarrow \Pi_B^{m+1} X$ is compressible into the fibrewise fat wedge $T_B^{m+1} X \subset \Pi_B^{m+1} X$ in $\underline{\mathcal{T}}_B^B$. Hence there is a map $\sigma : X \rightarrow P_B^m(\mathcal{L}_B^B(X))$ in $\underline{\mathcal{T}}_B^B$ such that $e_m^X \circ \sigma \sim_B 1_X$ in $\underline{\mathcal{T}}_B^B$. The converse is clear by the definition of $P_B^m(\mathcal{L}_B^B(X))$. \square

In the rest of this section, we work within the category $\underline{\mathcal{T}}_B$ of fibrewise *unpointed* spaces and maps between them. But we concentrate ourselves to consider its full subcategory $\underline{\mathcal{T}}_B^*$ of all fibrewise pointed spaces, so in $\underline{\mathcal{T}}_B^*$, we have more maps than in $\underline{\mathcal{T}}_B^B$ while we have just the same objects as in $\underline{\mathcal{T}}_B^B$.

Let X be a fibrewise pointed space over B . For given $k \geq 0$, we denote by $\Pi_B^{k+1} X$ the $(k+1)$ -fold fibrewise product and by $T_B^{k+1} X$ the $(k+1)$ -fold fibrewise fat wedge. Then $\text{cat}_B^*(X) \leq m$ if the $(m+1)$ -fold fibrewise diagonal map $\Delta_B^{m+1} : X \rightarrow \Pi_B^{m+1} X$ is compressible into the fibrewise fat wedge $T_B^{m+1} X$ in $\underline{\mathcal{T}}_B^*$. If there is no such m , we say $\text{cat}_B^*(X) = \infty$. Let us consider the case when $\text{cat}_B^*(X) < \infty$. The definition of a fibrewise A_∞ structure yields the following.

Theorem 7.2. *Let X be a fibrewise pointed space over B and $m \geq 0$. Then $\text{cat}_B^*(X) \leq m$ if and only if $\text{id}_X : X \rightarrow X$ has a lift to $P_B^m(\mathcal{L}_B^B(X)) \xrightarrow{e_m^X} X$ in the category $\underline{\mathcal{T}}_B^*$.*

Proof: If $\text{cat}_B^*(X) \leq m$, then the fibrewise diagonal $\Delta_B^{m+1} : X \rightarrow \prod_B^{m+1} X$ is compressible into the fibrewise fat wedge $\bigvee_B^{m+1} X \subset \prod_B^{m+1} X$ in $\underline{\mathcal{T}}_B^*$. Hence there is a map $\sigma : X \rightarrow P_B^m(\mathcal{L}_B^B(X))$ in $\underline{\mathcal{T}}_B^*$ such that $e_m^X \circ \sigma \sim_B 1_X$ in $\underline{\mathcal{T}}_B^*$. The converse is clear by the definition of $P_B^m(\mathcal{L}_B^B(X))$. \square

8. UPPER AND LOWER ESTIMATES

For X a fibrewise pointed space over B , we define a fibrewise version of Ganea's strong L-S category (see Ganea [6]) of X as $\text{Cat}_B^B(X)$ and also a fibrewise version of Fox's categorical length (see Fox [5] and Iwase [10]) of X as $\text{catlen}_B^B(X)$.

Definition 8.1. *Let X be a fibrewise pointed space over B .*

- (1) $\text{Cat}_B^B(X)$ is the least number $m \geq 0$ such that there exists a sequence $\{(X_i, h_i) \mid h_i : A_i \rightarrow X_{i-1}, 0 \leq i \leq m\}$ of pairs of space and map satisfying $X_0 = B$ and $X_m \simeq_B X$ in $\underline{\mathcal{T}}_B^B$ with the following homotopy push-out diagrams:

$$\begin{array}{ccc} A_i & \xrightarrow{p_{A_i}} & B \\ h_i \downarrow & & \downarrow s_{X_i} \\ X_{i-1} & \longrightarrow & X_i \end{array}$$

- (2) $\text{catlen}_B^B(X)$ is the least number $m \geq 0$ such that there exists a sequence $\{X_i \mid h_i : A_i \rightarrow X_{i-1}, 0 \leq i \leq m\}$ of spaces satisfying $X_0 = B$ and $X_m \simeq_B X$ in $\underline{\mathcal{T}}_B^B$ and that $\Delta_B : X_i \rightarrow X_i \times_B X_{i-1}$ is compressible into $X_i \times_B X_{i-1} \cup B \times_B X_i$ in $X_m \times_B X_m$.

A lower bound for the fibrewise L-S category of a fibrewise pointed space X over B can be described by a variant of cup length: since X is a fibrewise pointed space over B , there is a projection $p_X : X \rightarrow B$ with its section $s_X : B \rightarrow X$. Hence we can easily observe for any multiplicative cohomology theory h that

$$h^*(X) \cong h^*(B) \oplus h^*(X, B),$$

where we may identify $h^*(X, B)$ with the ideal $\ker s_X^* : h^*(X) \rightarrow h^*(B)$.

Definition 8.2. *For a fibrewise pointed space X over B and any multiplicative cohomology theory h , we define*

$$\begin{aligned} \text{cup}_B^B(X; h) &= \text{Max} \{m \geq 0 \mid \exists \{u_1, \dots, u_m \in h^*(X, B)\} \text{ s.t. } u_1 \cdots u_m \neq 0\}, \\ \text{cup}_B^B(X) &= \text{Max} \{\text{cup}_B^B(X; h) \mid h \text{ is a multiplicative cohomology theory}\}. \end{aligned}$$

We often denote $\text{cup}_B^B(\ ; h)$ by $\text{cup}_B^B(\ ; R)$ when $h^*(\) = H^*(\ ; R)$, where R is a ring with unit.

Let us recall that the relationship between an A_∞ -structure and a Lusternik-Schnirelmann category gives the key observation in [7, 8, 9].

On the other hand, Rudyak [17] and Strom [23] introduced a homotopy theoretical version of Fadell-Husseini's category weight, which can be translated into our setting as follows: for any fibrewise pointed space X over B , let $\{p_k^{\mathcal{L}_B^B(X)} : E_B^k(\mathcal{L}_B^B(X)) \rightarrow P_B^{k-1}(\mathcal{L}_B^B(X)); k \geq 1\}$ be the fibrewise A_∞ -structure of $\mathcal{L}_B^B(X)$ in the sense of Stasheff [22] (see also [11] for some more properties). Let h be a generalised cohomology theory.

Definition 8.3. For any $u \in h^*(X, B)$, we define

$$\text{wgt}_B^B(u; h) = \text{Min} \{m \geq 0 \mid (e_m^X)^*(u) \neq 0\},$$

where e_m^X is the composition of fibrewise maps $P_B^m(\mathcal{L}_B^B(X)) \hookrightarrow P_B^\infty(\mathcal{L}_B^B(X)) \xrightarrow[e_B]{e_\infty^X} X$.

Using this, we introduce some more invariants as follows.

Definition 8.4. For any fibrewise pointed space X over B , we define

$$\begin{aligned} \text{wgt}_\pi(X; h) &= \text{Max} \{ \text{wgt}_\pi(u; h) \mid u \in h^*(X, B) \}, \\ \text{wgt}_\pi(X) &= \text{Max} \{ \text{wgt}_\pi(X; h) \mid h \text{ is a generalised cohomology theory} \}, \\ \text{wgt}_B^B(X; h) &= \text{Max} \{ \text{wgt}_B^B(u; h) \mid u \in h^*(X, B) \}, \\ \text{wgt}_B^B(X) &= \text{Max} \{ \text{wgt}_B^B(X; h) \mid h \text{ is a generalised cohomology theory} \}. \end{aligned}$$

We often denote $\text{wgt}_\pi(\ ; h)$ and $\text{wgt}_B^B(\ ; h)$ by $\text{wgt}_\pi(\ ; R)$ and $\text{wgt}_B^B(\ ; R)$ respectively when $h^*(\) = H^*(\ ; R)$, where R is a ring with unit. We define versions of module weight for a fibrewise pointed space over B .

Definition 8.5. For a fibrewise pointed space X over B , we define

- (1) $\text{Mwgt}_B^B(X; h) = \text{Min} \left\{ m \geq 0 \mid (e_m^X)^* \text{ is a split mono of } (unstable) h^*h\text{-modules} \right\}$ for a generalised cohomology theory h .
- (2) $\text{Mwgt}_B^B(X) = \text{Max} \{ \text{Mwgt}_B^B(X; h) \mid h \text{ is a generalised cohomology theory} \}.$

Then we immediately obtain the following result.

Theorem 8.6. For any fibrewise pointed space X over B , we have

$$\text{cup}_B^B(X) \leq \text{wgt}_B^B(X) \leq \text{Mwgt}_B^B(X) \leq \text{cat}_B^B(X) \leq \text{catlen}_B^B(X) \leq \text{Cat}_B^B(X).$$

By Lemma 4.1, we have the following as a corollary of Theorem 1.13.

Corollary 8.7. For any space B having the homotopy type of a locally finite simplicial complex, we obtain

$$\mathcal{Z}_\pi(B) \leq \text{wgt}_\pi(B) \leq \text{Mwgt}_B^B(d(B)) \leq \mathcal{TC}(B) - 1 \leq \text{catlen}_B^B(d(B)) \leq \text{Cat}_B^B(d(B)).$$

9. HIGHER HOPF INVARIANTS

For any fibrewise pointed map $f : \mathcal{S}_B^B(V) \rightarrow X$ in $\underline{\mathcal{T}}_B^B$, we have its adjoint $\text{ad } f : V \rightarrow \mathcal{L}_B^B(X)$ such that

$$e_1^X \circ \mathcal{S}_B^B(\text{ad } f) = f : \mathcal{S}_B^B(V) \rightarrow X.$$

If $\text{cat}_B^B(X) \leq m$, then there is a fibrewise pointed map $\sigma : X \rightarrow P_B^m \mathcal{L}_B^B(X)$ in $\underline{\mathcal{T}}_B^B$ such that

$$e_1^X \circ \sigma \simeq_B^B \text{id}_X : X \rightarrow X.$$

Hence both the fibrewise maps $e_1^X \circ (\sigma \circ f)$ and $e_1^X \circ \mathcal{S}_B^B(\text{ad } f)$ are fibrewise pointed homotopic to f in $\underline{\underline{\mathcal{T}}}_B^B$. Then we have

$$e_1^X \circ \{\mathcal{S}_B^B(\text{ad } f) - (\sigma \circ f)\} \simeq_B^B *_B,$$

where \simeq_B^B denotes the fibrewise pointed homotopy and $*_B$ denotes the fibrewise trivial map in $\underline{\underline{\mathcal{T}}}_B^B$. Thus there is a fibrewise pointed map $H_m^\sigma(f) : \mathcal{S}_B^B(V) \rightarrow E_B^{m+1} \mathcal{L}_B^B(X)$ such that

$$p_m^{\mathcal{L}_B^B(X)} \circ H_m^\sigma(f) \simeq_B^B \mathcal{S}_B^B(\text{ad } f) - (\sigma \circ f).$$

Definition 9.1. Let X be of $\text{cat}_B^B(X) \leq m$, $m \geq 0$. For $f : \mathcal{S}_B^B(V) \rightarrow X$, we define

- (1) $H_m^B(f) = \{H_m^\sigma(f) | e_1^X \circ \sigma \simeq_B^B \text{id}_X\} \subset [\mathcal{S}_B^B(V), X],$
- (2) $\mathcal{H}_m^B(f) = \{(\mathcal{S}_B^B)_* H_m^\sigma(f) | e_1^X \circ \sigma \simeq_B^B \text{id}_X\} \subset \{\mathcal{S}_B^B(V), X\}_B^B,$

where, for two fibrewise spaces V and W , we denote by $\{V, W\}_B^B$ the homotopy set of fibrewise stable maps from V to W .

APPENDIX A. FIBREWISE HOMOTOPY PULL-BACKS AND PUSH-OUTS

In this paper, we are using A_∞ structures which is constructed using tools in $\underline{\underline{\mathcal{T}}}_B$ and $\underline{\underline{\mathcal{T}}}_B^B$ — especially, finite homotopy limits and colimits, in other words, fibrewise homotopy pull-backs and push-outs in $\underline{\underline{\mathcal{T}}}_B$ and $\underline{\underline{\mathcal{T}}}_B^B$. We show in this section that such constructions are possible even when a fibrewise space has some singular fibres.

First we consider the fibrewise homotopy pull-backs in $\underline{\underline{\mathcal{T}}}_B^B$: let X, Y, Z and E be fibrewise spaces over B and $p : E \rightarrow Z$ be a fibrewise fibration in $\underline{\underline{\mathcal{T}}}_B^B$. For any fibrewise map $f : X \rightarrow Z$ in $\underline{\underline{\mathcal{T}}}_B^B$, there exists a pull-back $X \xleftarrow{f^*p} f^*E \xrightarrow{\hat{f}} E$ of $X \xrightarrow{f} Z \xleftarrow{p} E$ as

$$f^*E = \{(x, e) \in X \times_B E | f(x) = p(e)\}$$

a subspace of $X \times_B E$ together with fibrewise maps $f^*p : f^*E \rightarrow X$ and $\hat{f} : f^*E \rightarrow E$ given by restricting canonical projections:

$$(f^*p)(x, e) = x, \quad \hat{f}(x, e) = e.$$

Theorem A.1 (Crabb-James [1]). *Let $p : E \rightarrow Z$ be a fibrewise fibration. For any fibrewise map $f : W \rightarrow Z$ in $\underline{\underline{\mathcal{T}}}_B^B$, $f^*p : f^*E \rightarrow W$ is also a fibrewise fibration.*

Let $\pi_t : \mathcal{P}_B(Z) \rightarrow Z$ be fibrewise fibrations given by $\pi_t(\ell) = \ell(t)$, $t = 0, 1$ (see also [1]). Then π_0 and π_1 induce a map $\pi : \mathcal{P}_B(Z) \rightarrow Z \times_B Z$ to the fibre product of two copies of $p_Z : Z \rightarrow B$.

Proposition A.2. $\pi : \mathcal{P}_B(Z) \rightarrow Z \times_B Z$ is a fibrewise fibration.

Proof: For any fibrewise map $\phi : W \rightarrow \mathcal{P}_B(Z)$ and a fibrewise homotopy $H : W \times [0, 1] = W \times_B (I_B) \rightarrow Z \times_B Z$ such that $H(w, 0) = \pi \circ \phi(w)$ for $w \in W$, we

define a fibrewise homotopy $\hat{H} : W \times [0, 1] = W \times_B (I_B) \rightarrow \mathcal{P}_B(Z) (\subset \mathcal{P}(Z))$ by

$$\hat{H}(w, s)(t) = \begin{cases} \text{pr}_0 \circ H(w, s), & \text{if } t = 0, \\ \text{pr}_0 \circ H(w, s-3t), & \text{if } 0 < t < \frac{s}{3}, \\ \pi_0 \circ \phi(w), & \text{if } t = \frac{s}{3}, \\ \phi(w)(\frac{3t-s}{3-2s}), & \text{if } \frac{s}{3} < t < \frac{3-s}{3}, \\ \pi_1 \circ \phi(w), & \text{if } t = \frac{3-s}{3}, \\ \text{pr}_1 \circ H(w, 3t-3+s), & \text{if } \frac{3-s}{3} < t < 1 \\ \text{pr}_1 \circ H(w, s), & \text{if } t = 1, \end{cases}$$

for $(w, s) \in W \times_B I_B$ and $t \in [0, 1]$, where $\text{pr}_k : Z \times_B Z \subset Z \times Z \rightarrow Z$ denotes the canonical projection given by $\text{pr}_k(z_0, z_1) = z_k$, $k = 0, 1$ for any $(z_0, z_1) \in Z \times_B Z$. Then for any $(w, s) \in W \times_B I_B$, we clearly have

$$\hat{H}(w, 0)(t) = \phi(w)(t), \quad t \in [0, 1],$$

$$(\hat{H}(w, s)(0), \hat{H}(w, s)(1)) = (\text{pr}_0 \circ H(w, s), \text{pr}_1 \circ H(w, s)) = H(w, s),$$

and hence we have $\hat{H}(w, 0) = \phi(w)$ for any $w \in W$ and also $\pi \circ \hat{H} = H$. This implies that \hat{H} is a fibrewise homotopy of ϕ covering H . Thus π is a fibrewise fibration. \square

This yields the following corollary.

Corollary A.3. *For any fibrewise maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ in $\underline{\mathcal{T}}_B$, the induced map $(f \times_B g)^* \pi : (f \times_B g)^* \mathcal{P}_B(Z) \rightarrow X \times_B Y$ is a fibrewise fibration in $\underline{\mathcal{T}}_B$.*

We often call the fibrewise space $(f \times_B g)^* \mathcal{P}_B(Z)$ together with the projections $\text{pr}_X \circ (f \times_B g)^* \pi : (f \times_B g)^* \mathcal{P}_B(Z) \rightarrow X$ and $\text{pr}_Y \circ (f \times_B g)^* \pi : (f \times_B g)^* \mathcal{P}_B(Z) \rightarrow Y$ the homotopy pull-back in $\underline{\mathcal{T}}_B$ of $X \xrightarrow{f} Z \xleftarrow{g} Y$. We remark that the above construction can be performed within $\underline{\mathcal{T}}_B^B$ if X, Y, Z, f and g are all in $\underline{\mathcal{T}}_B^B$, so that we have a pointed version of a fibrewise homotopy pull-back:

Corollary A.4. *For any fibrewise maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ in $\underline{\mathcal{T}}_B^B$, the induced map $(f \times_B g)^* \pi : (f \times_B g)^* \mathcal{P}_B(Z) \rightarrow X \times_B Y$ is a fibrewise fibration in $\underline{\mathcal{T}}_B^B$.*

Second we consider the fibrewise homotopy push-outs in $\underline{\mathcal{T}}_B^B$: let X, Y, Z and W be fibrewise pointed spaces over B and $i : Z \rightarrow W$ be a fibrewise cofibration in $\underline{\mathcal{T}}_B^B$.

For any fibrewise map $f : Z \rightarrow X$ over B , there exists a push-out $X \xrightarrow{f_* i} f_* W \xleftarrow{\check{f}} W$ of $X \xleftarrow{f} Z \xrightarrow{i} W$ as a quotient space of $X \amalg_B W$ by gluing $f(z)$ with $i(z)$ together with fibrewise maps $f_* i$ and \check{f} induced from the canonical inclusions.

Theorem A.5 (Crabb-James [1]). *Let $i : Z \rightarrow W$ be a fibrewise cofibration in $\underline{\mathcal{T}}_B$ (or $\underline{\mathcal{T}}_B^B$). For any fibrewise map $f : Z \rightarrow X$ in $\underline{\mathcal{T}}_B$ (or $\underline{\mathcal{T}}_B^B$, resp.), $f_* i : X \rightarrow f_* W$ is also a fibrewise cofibration in $\underline{\mathcal{T}}_B$ (or $\underline{\mathcal{T}}_B^B$, resp.).*

Let us recall that $\mathcal{I}_B^B(Z)$ is obtained from $\mathcal{I}_B(Z) = Z \times_B (B \times [0, 1]) = Z \times [0, 1]$ by identifying the subspace $s_Z(B) \times [0, 1] \subset Z \times [0, 1]$ with $s_Z(B)$ by the canonical projection to the first factor : $s_Z(B) \times [0, 1] \rightarrow s_Z(B)$. Let $\iota_t : Z \rightarrow \mathcal{I}_B^B(Z)$ be fibrewise cofibration in $\underline{\mathcal{T}}_B^B$ given by $\iota_t(z) = q(z, t)$, $0 \leq t \leq 1$, where $q : Z \times [0, 1] \rightarrow \mathcal{I}_B^B(Z)$ denotes the identification map. Then ι_0 and ι_1 induce a map $\iota : Z \vee_B Z \rightarrow \mathcal{I}_B^B(Z)$ from $Z \vee_B Z$ the push-out of two copies of $s_Z : B \rightarrow Z$.

Proposition A.6. $\iota : Z \vee_B Z \rightarrow \mathcal{I}_B^B(Z)$ is a fibrewise cofibration.

Proof: For any fibrewise map $\phi : \mathcal{I}_B^B(Z) \rightarrow W$ and a fibrewise homotopy $H : (Z \vee_B Z) \times [0, 1] = (Z \vee_B Z) \times_B I_B \rightarrow W$ such that $H(z, 0) = \phi \circ \iota(z)$ for $z \in Z \vee_B Z$, we define a fibrewise homotopy $\check{H} : \mathcal{I}_B^B(Z) \times [0, 1] = \mathcal{I}_B^B(Z) \times_B (I_B) \rightarrow W$ by

$$\check{H}(q(z, t), s) = \begin{cases} H(\text{in}_0(z), s-3t), & \text{if } 0 \leq t < \frac{s}{3}, \\ \phi \circ \iota_0(z), & \text{if } t = \frac{s}{3}, \\ \phi(q(z, \frac{3t-s}{3-2s})), & \text{if } \frac{s}{3} < t < \frac{3-s}{3}, \\ \phi \circ \iota_1(z), & \text{if } t = \frac{3-s}{3}, \\ H(\text{in}_1(z), 3t-3+s), & \text{if } \frac{3-s}{3} < t \leq 1 \end{cases}$$

for $(q(z, t), s) \in \mathcal{I}_B^B(Z) \times_B I_B$, where $\text{in}_k : Z \hookrightarrow Z \vee_B Z$, $k = 0, 1$ denote the canonical inclusion given by $\text{in}_0(z) = (z, *_b)$ and $\text{in}_1(z) = (*_b, z)$, $b = p_Z(z)$ for any $z \in Z$. Then for any $(q(z, t), s) \in \mathcal{I}_B^B(Z) \times_B I_B$, we clearly have

$$\begin{aligned} \check{H}(q(z, t))(0) &= \phi(q(z, t)), \\ \check{H}(q(z, 0))(s) &= H(\text{in}_0(z), s), \quad \check{H}(q(z, 1))(s) = H(\text{in}_1(z), s), \end{aligned}$$

and hence we have $\check{H}(q(z, t))(0) = \phi(q(z, t))$ for any $q(z, t) \in \mathcal{I}_B^B(Z)$ and also $\check{H} \circ (\iota \times_B 1_{I_B}) = H$. This implies that \check{H} is a fibrewise homotopy of ϕ extending H . Thus ι is a fibrewise cofibration. \square

This yields the following corollary.

Corollary A.7. For any fibrewise maps $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ in $\underline{\mathcal{T}}_B^B$, the induced map $(f \vee_B g)_* \iota : X \vee_B Y \rightarrow (f \vee_B g)^* \mathcal{I}_B^B(Z)$ is a fibrewise cofibration in $\underline{\mathcal{T}}_B^B$.

We often call the fibrewise space $(f \vee_B g)^* \mathcal{I}_B^B(Z)$ together with the inclusions $(f \vee_B g)_* \iota \circ \text{in}_X : X \rightarrow (f \vee_B g)^* \mathcal{I}_B^B(Z)$ and $(f \vee_B g)_* \iota \circ \text{in}_Y : Y \rightarrow (f \vee_B g)^* \mathcal{I}_B^B(Z)$ as homotopy push-out in $\underline{\mathcal{T}}_B^B$ of $X \xleftarrow{f} Z \xrightarrow{g} Y$.

Quite similarly for a fibrewise space Z in $\underline{\mathcal{T}}_B$, we obtain a fibrewise cofibration $\hat{\iota} : Z \amalg Z = Z \times \{0\} \cup Z \times \{1\} \hookrightarrow Z \times [0, 1] = \mathcal{I}_B(Z)$. Thus we have the following.

Corollary A.8. For any fibrewise maps $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ in $\underline{\mathcal{T}}_B$, the induced map $(f \amalg g)_* \hat{\iota} : X \amalg Y \rightarrow (f \amalg g)^* \mathcal{I}_B(Z)$ is a fibrewise cofibration in $\underline{\mathcal{T}}_B$.

Thus we also have an unpointed version of a fibrewise homotopy push-out.

REFERENCES

- [1] M. C. Crabb. and I. M. James, “Fibrewise Homotopy Theory”, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 1998.
- [2] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003), 211–221.
- [3] M. Farber, *Topology of robot motion planning*, “Morse theoretic methods in nonlinear analysis and in symplectic topology”, 185–230, NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006.
- [4] M. Farber and M. Grant, *Symmetric Motion Planning*, Topology and robotics, 85–104, Contemp. Math., 438, Amer. Math. Soc., Providence, RI, 2007.
- [5] R. H. Fox, *On the Lusternik-Schnirelmann category*, Ann. of Math. (2) **42**, (1941), 333–370.
- [6] T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois. J. Math. **11** (1967), 417–427.

- [7] N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. Lon. Math. Soc., **30** (1998), 623–634.
- [8] N. Iwase, *A_∞ -method in Lusternik-Schnirelmann category*, Topology **41** (2002), 695–723.
- [9] N. Iwase, *Lusternik-Schnirelmann category of a sphere-bundle over a sphere*, Topology **42** (2003), 701–713.
- [10] N. Iwase, *Categorical length, relative L-S category and higher Hopf invariants*, preprint.
- [11] N. Iwase and M. Mimura, *Higher homotopy associativity*, Algebraic Topology, (Arcata CA 1986), Lect. Notes in Math. **1370**, Springer Verlag, Berlin (1989) 193–220.
- [12] N. Iwase and M. Sakai, *Functors on the category of quasi-fibrations*, Topology Appl. **155** (2008), 1403–1409.
- [13] I. M. James and J. R. Morris, *Fibrewise category*, Proc. Roy. Soc. Edinburgh. **119A** (1991), 177–190.
- [14] I. M. James, *Introduction to fibrewise homotopy theory*, “Handbook of algebraic topology”, 169–194, North Holland, Amsterdam, 1995.
- [15] I. M. James, *Lusternik-Schnirelmann Category*, “Handbook of algebraic topology”, 1293–1310, North Holland, Amsterdam, 1995.
- [16] J. Milnor, *On Spaces Having the Homotopy Type of a CW-Complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.
- [17] Y. B. Rudyak, *On category weight and its applications*, Topology **38** (1999), 37–55.
- [18] Y. B. Rudyak, *On analytical applications of stable homotopy (the Arnold conjecture, critical points)*, Math. Z. **230**(1999) 659–672.
- [19] M. Sakai, *The functor on the category of quasi-fibrations*, DSc Thesis (Kyushu University 1999), 1999.
- [20] M. Sakai, *A proof of the homotopy push-out and pull-back lemma*, Proc. Amer. Math. Soc. **129** (2001), 2461–2466.
- [21] A. S. Schwarz *The genus of a fiber space*, Amer. Math. Soc. Transl.(2) **55** (1966), 49–140.
- [22] J. D. Stasheff, *Homotopy associativity of H-spaces, I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312.
- [23] J. Strom, *Essential category weight and phantom maps*, Cohomological methods in homotopy theory (Bellaterra, 1998), 409–415, Progr. Math., 196, Birkhauser, Basel, 2001.
- [24] G. W. Whitehead, “Elements of Homotopy Theory”, Springer Verlag, Berlin, GTM series **61**, 1978.

E-mail address: iwase@math.kyushu-u.ac.jp

E-mail address, Sakai: sakai@kurume-nct.ac.jp

(Iwase) FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, FUKUOKA 810-8560, JAPAN

(Sakai) KURUME NATIONAL COLLEGE OF TECHNOLOGY, FUKUOKA 830-8555, JAPAN.